An introduction to SMT solving with quantifiers

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Agenda

- What are quantifiers for in SMT?
- An overview of classic instantiation techniques
  - Trigger-based instantiation
  - Conflict-based instantiation
  - Model-based instantiation
- A unifying framework for classic instantiation techniques
- Effective enumerative instantiation
- Playing with different instantiation techniques

Why do we need quantifiers?

- We've just seen how powerful and flexible SMT solvers are.
- The efficiency of SMT solvers comes from dedicated decision procedures for their theories.
- But what if the problem you want to solve does not fit existing theories?

Example

```
(set-logic UF)

(declare-sort U 0)
(declare-fun f (U U) U)
(declare-const a U)
(declare-const b U)

(assert (not (= (f a b) (f b a))))



(check-sat)
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(assert (forall ((x U) (y U)) (= (f x y) (f y x))))

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Quantifiers are important for many applications

- Automatic theorem proving
  - Adding axioms for new symbols (tools such Sledghammer [BKPU16])
- Software verification
  - Encoding contracts (tools such as Dafny [Lei10] and Verus [LHC+23] rely heavily on quantifiers)
- Function synthesis
  - Specifying the behavior of a function to synthesize [ABJ+13; RBN+19]

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  - Undecidable in general
  - Explosive heuristics
  - Users want it to work as well as on quantifier-free problems

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- But as we will see today, state-of-the-art solvers do well with quantifiers in practice

Satisfiability Modulo Theories (SMT)

First-order formulas in CNF:

$$\begin{array}{lll} t & ::= & x \mid f(t, \dots, t) \\ \varphi & ::= & p(t, \dots, t) \mid \neg\varphi \mid \varphi \vee \varphi \mid \forall x_1 \dots x_n .\, \varphi \end{array}$$

Given a formula φ in FOL and background theories $\mathcal{T}_1, \dots, \mathcal{T}_n$, finding a model $\mathcal{M}$ giving an *interpretation* to all terms and predicates such that $\mathcal{M} \models_{\mathcal{T}_1, \dots, \mathcal{T}_n} \varphi$.

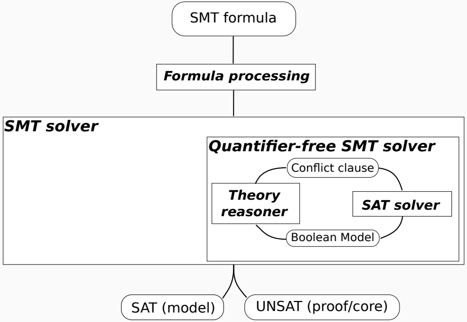
- Quantified formulas can be classified as *strong* and *weak* quantifiers, which means to occur in a negative (e.g., under a single negation) or positive context.
- It is sound to Skolemize strong quantifiers:

$$\frac{\exists x .\, \varphi[x]}{\varphi[k]} \text{ where } k \text{ is a fresh function symbol}$$

- If Skolemization is done under other quantifiers, the introduced function must take the respective quantified variables as arguments.

$$\frac{\forall y .\, \exists x .\, p(x, y)}{\forall y .\, p(f(y), y)}$$

CDCL(T) architecture



Quantifier-free solver enumerates models E

- E is a set of ground literals $\{a \leq b,\ b \leq a + x,\ x \simeq 0,\ f(a) \not\simeq f(b)\}$

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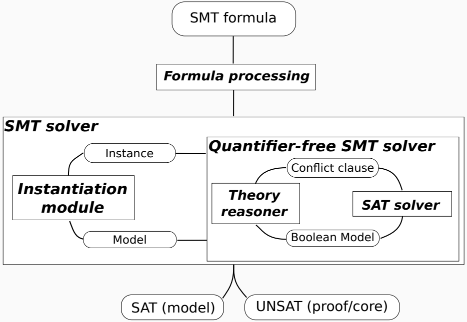

Quantifier-free solver enumerates models $E \cup \mathrm{Q}$

- E is a set of ground literals $\{a \leq b,\ b \leq a + x,\ x \simeq 0,\ f(a) \not\simeq f(b)\}$
- Q is a set of quantified clauses $\{\forall xyz.\ f(x) \not\simeq f(z) \vee g(y) \simeq h(z)\}$

Instantiation module generates instances of Q $f(a) \not\simeq f(b) \vee g(a) \simeq h(b)$

The abstract procedure: ground case

```
function CheckSat(φ, 𝒯) is
  φ ← Process(φ)                          // Simplifications, CNF transformation
  do
    E ← CheckBoolean(abs(φ))              // SAT solver
    if E = ∅ then
      return Unsat
    C ← CheckGround(E, 𝒯)                 // Theory solvers
    φ ← φ ∪ C
  while C ≠ ∅
  return Sat
```

The abstract procedure: quantified case

```
function CheckSatQ(φ, 𝒯) is
  φ ← Process(φ)                              // Simplifications, CNF transformation
  do
    ⟨E, 𝒬⟩ ← CheckBoolean(absQ(φ))            // SAT solver
    if E ∪ 𝒬 = ∅ then
      return Unsat
    C ← CheckGround(E, 𝒯)                     // Theory solvers
    if C ≠ ∅ then
      φ ← φ ∪ C
      continue
    ℐ ← Inst(E, 𝒬, 𝒯)                         // Instantiation module
    φ ← φ ∪ ℐ
  while ℐ ≠ ∅
  if models can be built for 𝒯 then
    return Sat
  else
    return Unknown
```

Why can we use instantiation?

Theorem (Herbrand)

A set of pure first-order logic formulas is unsatisfiable if and only if there exists a finite unsatisfiable set of its instances.

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All humans are mortal
All Greeks are humans

Then all Greeks are mortal

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  $\Big((\forall x.\, H(x) \rightarrow M(x)) \wedge (\forall x.\, G(x) \rightarrow H(x))\Big) \rightarrow \forall x.\, G(x) \rightarrow M(x)$

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- A ground SMT solver will deduce unsatisfiability.

Instantiation is not the only way to reason about first-order logic

- Superposition-based, tableaux-based systems are well-established theorem provers
  - Vampire [KV13], E [SCV19], ...
  - Princess [BR15], LeanCop [Ott08], ...

- The focus on instantiation in SMT can be explained by how it makes “quantifier reasoning” simulate how the other theory solvers work, which is well-suited for the CDCL(T) architecture.

Instantiation techniques

- Trigger-based [DNS05; MB07]
- Conflict-based [RTM14; BFR17]
- Model-based [GM09; RTG+13]

⊕ General: ∀+EUF+...

⊖ Finding instantiations is hard

- Enumerative [RBF18]

⊕ Easy to implement

⊕ Reliable last resort

- QE-based [Mon10; Bjø10; RDK+15; BJ15]

⊕ Decision procedures available

⊖ Pure fragments

- Syntax-Guided Synthesis (SyGuS)-based [NPR+21]

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Instantiation strategies: trigger-based

[Detlefs et al. J. ACM'05]

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Trigger-based instantiation (E-matching): search for relevant instantiations according to a set of triggers and E-matching

- A *trigger* is a set of terms whose free variables should cover the respective quantified variables.

- E $= \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and Q $= \{\forall x.\, P(x) \vee R(x)\}$
- Assume the trigger $\{(P(x))\}$.
- Since E $\models P(x)\{x \mapsto t\} \simeq P(t)$, for $t = a, b, c$, this strategy may return $\{\{x \mapsto a\},\ \{x \mapsto b\},\ \{x \mapsto c\}\}$.
- Formally:

$\mathbf{e}$(E, $\forall \bar{x}.\, \varphi$):

1. Select a trigger $\{\bar{t}_1, \ldots \bar{t}_n\}$ for $\forall \bar{x}.\, \varphi$.
2. For each $i = 1, \ldots, n$, select a set of substitutions S_i s.t. for each $\sigma \in S_i$, E $\models \bar{t}_i\sigma \simeq \bar{g}_i$ for some tuple $\bar{g}_i \in \mathbf{T}$(E).
3. Return $\bigcup_{i=1}^{n} S_i$.

Trigger-based instantiation is highly dependent on the chosen triggers

- A proper selection of triggers may guarantee a decision procedure for some fragments [DCKP13].
- But in general, trigger selection can have a high impact on the solver's success rate
- Again for E $= \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and Q $= \{\forall x.\, P(x) \vee R(x)\}$
- Assume the trigger $\{(R(x))\}$.
- Now there is only one possible instantiation: $\{x \mapsto b\}$

A trigger selection strategy [LP16]

1. Traverse formula and collect *trigger heads* and *trigger killers*.
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3. For the formula $\forall x.\, p_1(x) \vee \cdots \vee p_n(x)$ we have:
   - Trigger heads: $\{p_1(x), \ldots, p_n(x)\}$
   - 2^n adequate candidates
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4. Remaining candidates ordered by specificity:
   - T_1 is less *specific* than T_2 if and only if all matchings of T_2 are also matchings for T_1
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5. Finally, the possible triggers for the quantified formula are the minimal candidates.

Trigger-based instantiation issues: matching loops

- A well known issue is matching loops: terms from previous instantiation rounds leading to more instantiations indefinitely

- consider E $= \{a \simeq f(a), \dots\}$ and Q $= \{\forall x.\, f(f(x)) \simeq f(x)\}$. What happens if the trigger is $\{f(x)\}$?

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- Some approaches introduced to address this issue, but they have limited application:
  - Select triggers such that they are not subterms of other terms in the formula [LP16]
  - Ignore during instantiation terms from instances that did not lead to conflicts [Bar16; MB07]

Trigger-based instantiation can be explosive

Pattern-matching of terms from Q into terms of E

- for $\forall xyz.\ f(x) \not\simeq f(z) \vee g(y) \simeq h(z)$ a trigger is $\{f(x),\ g(y),\ h(z)\}$

⊖ Multi-term triggers specially can lead to many instantiations

E with 10^2 applications each for f, g, h leads to up to 10^6 instantiations

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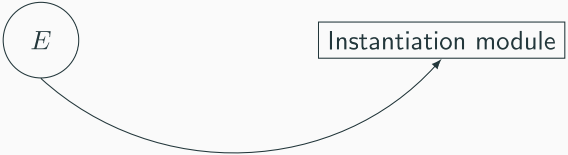

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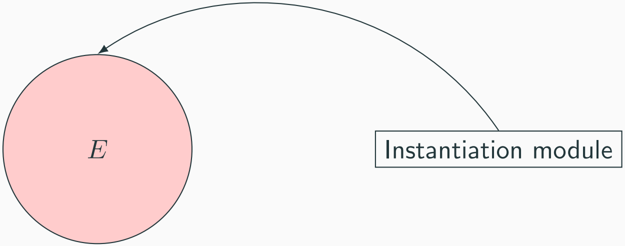

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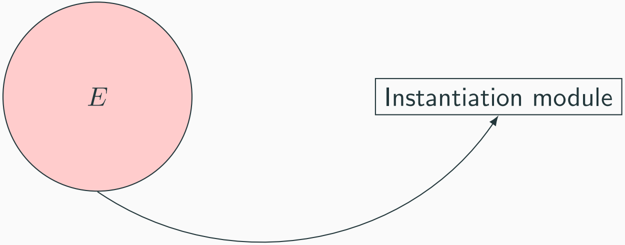

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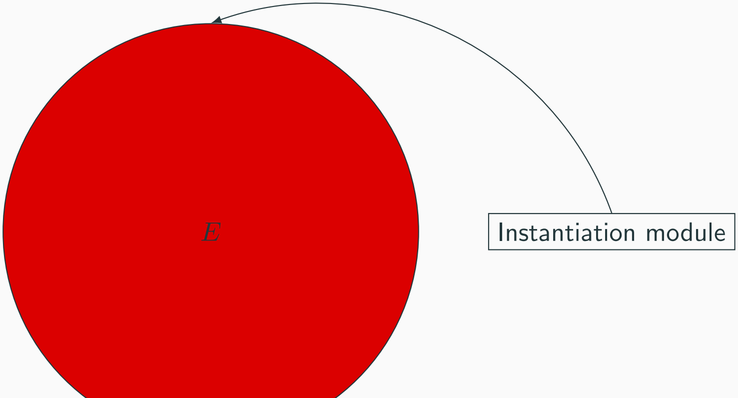

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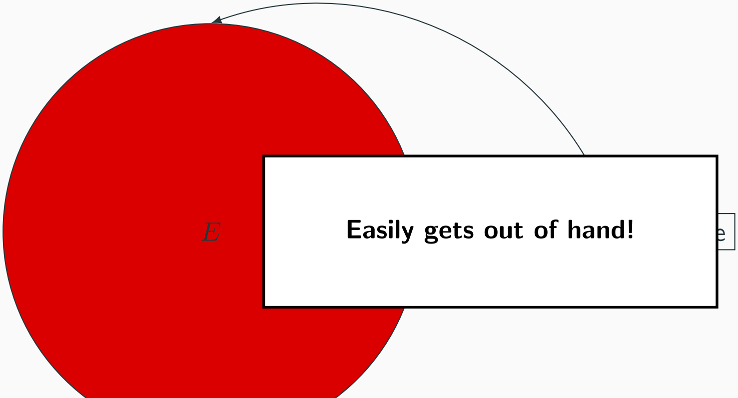

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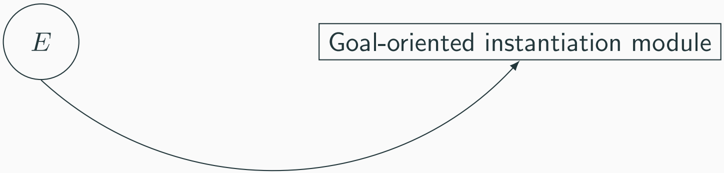
Goal-oriented instantiation

Check consistency of $E \cup Q$

⊕ Only instances refuting the current model are generated

If $\{f(a) \simeq f(c),\, g(b) \not\simeq h(c)\} \subseteq E$, then E is refuted with the instantiation

$\forall xyz.\, f(x) \not\simeq f(z) \vee g(y) \simeq h(z) \rightarrow f(a) \not\simeq f(c) \vee g(b) \simeq h(c)$



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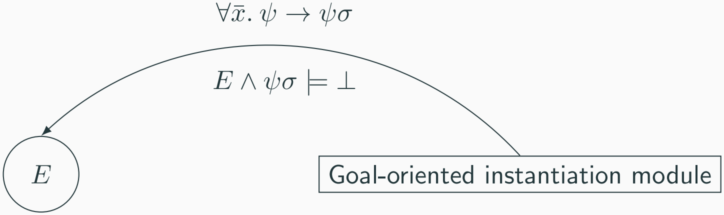

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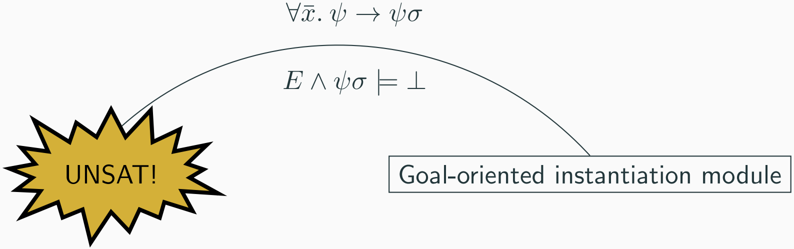
  If $\{f(a) \simeq f(c),\, g(b) \not\simeq h(c)\} \subseteq E$, then E is refuted with the instantiation

  $\forall xyz.\, f(x) \not\simeq f(z) \lor g(y) \simeq h(z) \rightarrow f(a) \not\simeq f(c) \lor g(b) \simeq h(c)$



Instantiation strategies: conflict-based [Reynolds et al. FMCAD'14]

Conflict-based instantiation: search for instantiations of a quantified formula in Q that make E unsatisfiable

Instantiation strategies: conflict-based

[Reynolds et al. FMCAD'14]

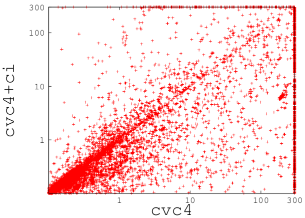
Conflict-based instantiation: search for instantiations of a quantified formula in Q that make E unsatisfiable

- E $= \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and Q $= \{\forall x.\, P(x) \vee R(x)\}$
- Since E, $P(b) \vee R(b) \models \bot$, this strategy will return $\{\{x \mapsto b\}\}$.
- Formally:

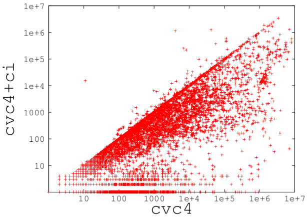
$\mathbf{c}$(E, $\forall \bar{x}.\, \varphi$): 1. Either return $\{\sigma\}$ where E, $\varphi\sigma \models \bot$, or return $\emptyset$.

Impact of conflict-based instantiation

[Reynolds et al. FMCAD'14]

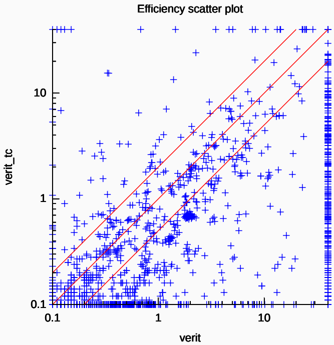


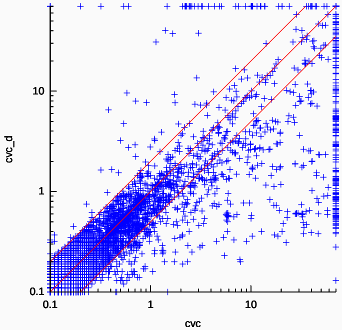
(a) Runtime (in seconds).



(b) Reported number of instances.

* experiments across 12468 benchmarks from TPTP, SMT-LIB, and Sledghammer.





veriT: $+$ 800 out of $1\,785$ unsolved problems

CVC4: $+$ 200 out of 745 unsolved problems

- Improvements on CVC4 came from discarding from trigger-based strategy instances already entailed by the formula: if $E \models \varphi[\bar{t}]$, for $\forall x.\, \varphi[x]$.

* experiments in the "UF", "UFLIA", "UFLRA" and "UFIDL" categories of SMT-LIB, which have $10\,495$ benchmarks annotated as unsatisfiable, with 30s timeout.

Caveats of conflct-based instantiation

- The search for the instantiations in practice is more expensive

- The technique is inherently incomplete: it only considers a single formula at a time
  - $E = \{p(a)\}$ and Q $= \{\forall x.\, q(x), \forall yz.\, \neg q(y) \vee \neg p(z)\}$.
    There are no substitutions σ, ρ such that $E \models \neg q(x)\sigma$ or $E \models q(y)\rho \wedge p(z)\rho$, even though $E \cup$ Q is clearly inconsistent.

- It should be seen as a *complement* to other techniques that are more general

Instantiation strategies: model-based

[Ge and de Moura CAV'09]

Model-based instantiation (MBQI): build a candidate model for $E \cup Q$ and instantiate with counter-examples from model checking

Instantiation strategies: model-based

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Model-based instantiation (MBQI): build a candidate model for $\mathsf{E} \cup \mathsf{Q}$ and instantiate with counter-examples from model checking

- $\mathsf{E} = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $\mathsf{Q} = \{\forall x.\, P(x) \vee R(x)\}$
- Assume that $P^{\mathcal{M}} = \lambda x.\, \mathsf{ite}(x \simeq c,\ \top,\ \bot)$ and $R^{\mathcal{M}} = \lambda x.\, \bot$.
- Since $\mathcal{M} \not\models P(a) \vee R(a)$, this strategy may return $\{\{x \mapsto a\}\}$.
- Formally:

$\mathbf{m}(\mathsf{E}, \forall \bar{x}.\, \varphi)$: 1. Construct a model $\mathcal{M}$ for E.

2. Return $\{\{\bar{x} \mapsto \bar{t}\}\}$ where $\bar{t} \in \mathbf{T}(\mathsf{E})$ and $\mathcal{M} \not\models \varphi\{\bar{x} \mapsto \bar{t}\}$, or $\emptyset$ if none exists.

Instantiation strategies: model-based

[Ge and de Moura CAV'09]

- MBQI is complete for a number of fragments
  - Bernays-Schönfinkel
  - *Essentially Uninterpreted Formulas*: "theory variables" only appear as arguments of uninterpreted functions
- It is a good strategy to complement incomplete techniques
- One should note that by its nature it is generally more successful on satisfiable problems

A unifying framework for classic instantiation techniques

Let's look deeper into the problem (with $T =$ EUF)

$E \models \neg\psi\sigma$, for some $\forall \bar{x}.\ \psi \in \mathsf{Q}$

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$$f(a) \simeq f(c) \wedge g(b) \not\simeq h(c) \models (f(x) \simeq f(z) \wedge g(y) \not\simeq h(z))\,\sigma$$

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$$\sigma = \{x \mapsto c,\ y \mapsto b,\ z \mapsto c\}$$

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or

$$\sigma = \{x \mapsto a,\ y \mapsto b,\ z \mapsto c\}$$

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E-ground (dis)unification [BFR17]

Given conjunctive sets of equality literals E and L, with E ground, finding a substitution σ s.t. $E \models L\sigma$

- Solution space can be restricted into ground terms from $E \cup L$
- NP-complete
  - NP: solutions can checked in polynomial time
  - NP-hard: reduction of 3-SAT into the entailment
- Variant of classic (non-simultaneous) rigid E-unification

$$s_1\sigma \simeq t_1\sigma, \ldots, s_n\sigma \simeq t_n\sigma \models u\sigma \simeq v\sigma$$

Casting instantiation techniques: Trigger-based

$$E \models (u_1 \simeq y_1 \wedge \cdots \wedge u_m \simeq y_m)\,\sigma$$

where $\{u_1, \ldots, u_m\}$ is a trigger for $\forall \bar{x}.\, \psi \in$ Q and each $y_i\sigma \in \mathbf{T}(E)$

- - $E = \{f(a) \simeq g(b),\ h(a) \simeq b,\ f(a) \simeq f(c)\}$
  - Q $= \{\forall x.\ f(x) \not\simeq g(h(x))\}$, Trigger$=\{f(x)\}$
- Solving $E \models (f(x) \simeq y)\sigma$ yields
  - $\sigma_1 = \{y \mapsto f(a),\, x \mapsto a\}$
  - $\sigma_2 = \{y \mapsto f(c),\, x \mapsto c\}$
- The instantiation lemmas are:
  - $\forall x.\ f(x) \not\simeq g(h(x)) \rightarrow f(a) \not\simeq g(h(a))$
  - $\forall x.\ f(x) \not\simeq g(h(x)) \rightarrow f(c) \not\simeq g(h(c))$

Casting instantiation techniques: Conflict-based

$$E \models \neg\psi\sigma, \text{ for some } \forall\bar{x}.\,\psi \in \mathsf{Q}$$

- Consider
  - $E = \{f(a) \simeq g(b),\ h(a) \simeq b,\ f(a) \simeq f(c)\}$
  - $\mathsf{Q} = \{\forall x.\ f(x) \not\simeq g(h(x))\}$
- Solving $E \models (f(x) \simeq g(h(x)))\sigma$ yields
  - $\sigma = \{x \mapsto a\}$
- The instantiation lemma is:
  - $\forall x.\ f(x) \not\simeq g(h(x)) \rightarrow f(a) \not\simeq g(h(a))$

Casting instantiation techniques: Model-based

$$E_{\text{TOT}} \models \neg\psi\sigma, \text{ for some } \forall\bar{x}.\,\psi \in \mathsf{Q}$$

where E_{TOT} is a total extension of E s.t.:

- ground terms not in E necessary for evaluating Q are added
- all terms in $\mathbf{T}(E)$ not asserted equal are made disequal

- Consider
  - $E = \{f(a) \simeq g(b),\ h(a) \simeq b\}$
  - $\mathsf{Q} = \{\forall x.\ f(x) \not\simeq g(x),\ \forall xy.\ \psi\}$, $e = a$ as a default value, and

$$\begin{aligned} E_{\text{TOT}} &= E \cup \quad \{a \not\simeq b, a \not\simeq f(a), b \not\simeq f(a)\} \\ &\qquad\ \cup \quad \{f(b) \simeq f(a), f(f(a)) \simeq f(a), g(a) \simeq a, g(f(a)) \simeq a\} \cup \{\dots\} \end{aligned}$$

- Solving $\{\dots, f(a) \simeq g(b), f(b) \simeq f(a), \dots\} \models f(x) \simeq g(x)\sigma$ yields
  - $\sigma = \{x \mapsto b\}$
- The lemma $\forall x.\ f(x) \not\simeq g(x) \rightarrow f(a) \not\simeq g(a)$ prevents the same E_{TOT}

How to solve the E-ground (dis)unification problem?

Entailment conditions:

- $E \models (x \simeq y)\sigma$
  - $x\sigma = y\sigma$ or
  - some t_1, t_2 s.t. $x\sigma \in [t_1]$, $y\sigma \in [t_2]$, and $[t_1] = [t_2]$

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- $E \models (x \simeq f(s_1, \dots, s_n))\sigma$, x occurs in $f(s_1, \dots, s_n)$,
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- $E \models (f(u_1, \ldots, u_n) \simeq g(v_1, \ldots, v_n))\sigma$ and
  - $f = g$ and $E \models u_1\sigma \simeq v_1\sigma$, ..., $E \models u_n\sigma \simeq v_n\sigma$ or
  - some t_1, $t_2 \in \mathbf{T}(E)$ s.t. $[t_1] = [t_2]$, $f(u_1, \ldots, u_n)\sigma \in [t_1]$, and $g(v_1, \ldots, v_n)\sigma \in [t_2]$

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- $E \models (u \not\simeq v)\sigma$
  - some t_1, $t_2 \in \mathbf{T}(E)$ s.t. $E \models t_1 \not\simeq t_2$, $u\sigma \in [t_1]$, and $v\sigma \in [t_2]$

Congruence Closure with Free Variables

Congruence Closure with Free Variables (CCFV) is a sound, complete and terminating calculus for solving E-ground (dis)unification

⊕ (allows for) Goal-oriented instantiation technique

⊕ Efficient

⊖ ~~Ad-hoc~~ **Versatile framework, recasting many instantiation techniques as a CCFV problem**

Finding solutions σ for $E \models L\sigma$

$$\begin{array}{rcl} E & \models & L\sigma \\ f(a) \simeq f(c) \wedge g(b) \not\simeq h(c) & \models & (f(x) \simeq f(z) \wedge g(y) \not\simeq h(z))\,\sigma \end{array}$$

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$$f(x) \simeq f(z) \wedge g(y) \not\simeq h(z)$$

$\emptyset$

$$f(x) \simeq f(z) \wedge z \simeq c \wedge y \simeq b$$

$y \simeq b$

$$f(x) \simeq f(z) \wedge z \simeq c$$

$y \simeq b,\ z \simeq c$

$$f(x) \simeq f(c)$$

$x \simeq a$ | $x \simeq c$

$x \simeq a,\ y \simeq b,\ z \simeq c$ | $x \simeq c,\ y \simeq b,\ z \simeq c$

Finding solutions σ for $E \models L\sigma$

$$
\begin{array}{rcl}
E & \models & L\sigma \\
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\end{array}
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$y \simeq b,\ z \simeq c$

$$f(x) \simeq f(c)$$

$x \simeq a$ | $x \simeq c$

$x \simeq a,\ y \simeq b,\ z \simeq c$ | $x \simeq c,\ y \simeq b,\ z \simeq c$

Finding solutions σ for $E \models L\sigma$

$$\begin{array}{rcl} E & \models & L\sigma \\ f(a) \simeq f(c) \land g(b) \not\simeq h(c) & \models & (f(x) \simeq f(z) \land g(y) \not\simeq h(z))\,\sigma \end{array}$$

$$f(x) \simeq f(z) \land g(y) \not\simeq h(z)$$

$\varnothing$

$$f(x) \simeq f(z) \land z \simeq c \land y \simeq b$$

$y \simeq b$

$$f(x) \simeq f(z) \land z \simeq c$$

$y \simeq b,\ z \simeq c$

$$f(x) \simeq f(c)$$

$x \simeq a$ | $x \simeq c$

$x \simeq a,\ y \simeq b,\ z \simeq c$ | $x \simeq c,\ y \simeq b,\ z \simeq c$

Finding solutions σ for $E \models L\sigma$

$$\begin{aligned} E \quad &\models \quad L\sigma \\ f(a) \simeq f(c) \wedge g(b) \not\simeq h(c) \quad &\models \quad (f(x) \simeq f(z) \wedge g(y) \not\simeq h(z))\,\sigma \end{aligned}$$

$$f(x) \simeq f(z) \wedge g(y) \not\simeq h(z)$$

$\varnothing$

$$f(x) \simeq f(z) \wedge z \simeq c \wedge y \simeq b$$

$y \simeq b$

$$f(x) \simeq f(z) \wedge z \simeq c$$

$y \simeq b,\ z \simeq c$

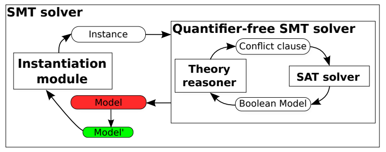
$$f(x) \simeq f(c)$$

$$x \simeq a \qquad\qquad x \simeq c$$

$x \simeq a,\ y \simeq b,\ z \simeq c$ | $x \simeq c,\ y \simeq b,\ z \simeq c$

$$\top \qquad\qquad \top$$

Implementation



- Model minimization
  - Relevancy
  - Prime implicant
- Top symbol indexing of E-graph from ground congruence closure

$$f \to \begin{cases} f([t_1], \ldots, [t_n]) \\ \ldots \\ f([t'_1], \ldots, [t'_n]) \end{cases}$$

  - $E \models f(x)\sigma \simeq t$ only if $[t]$ contains some $f(t')$
    $E \models f(x)\sigma \simeq g(y)\sigma$ only if some $[t]$ contains some $f(t')$ and some $g(t'')$
    - Bitmasks for fast checking if symbol has applications in congruence class
- Mapping from congruence class to classes it's disequal to

Implementation

- Selection strategies

$$E \models f(x, y) \simeq h(z) \land x \simeq t \land \dots$$

- Eagerly checking whether constraints can be discarded
  - After assigning x to t, the remaining problem is normalized

$$E \models f(t, y) \simeq h(z) \land \dots$$

  - $E \models f(t, y)\sigma \simeq h(z)\sigma$ only if there is some $f(t', t'')$ s.t.

$$E \models t \simeq t'$$

Effective enumerative instantiation

Instantiation and the Herbrand Theorem

- The earliest theorem provers relied on *Herbrand instantiation*
  - Instantiate with all possible terms in the Herbrand universe (all possible well-sorted ground terms in the formula's signature)
- Enumerating all instances is unfeasible in practice!
- Enumerative instantiation was then discarded

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But enumerative instantiation can be effective for state-of-the-art SMT

- strengthened version of the Herbrand theorem
- efficient implementation techniques

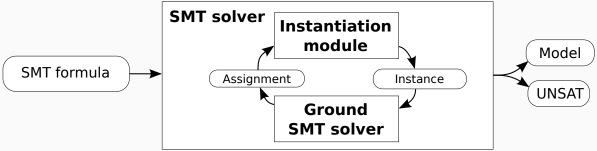
Theorem (Strengthened Herbrand [Kan63; RBF18])

If there exists an infinite series of finite satisfiable sets of ground literals E_i *and of finite sets of ground instances* Q_i *of* Q *such that*

- $\mathsf{Q}_i = \{\varphi\sigma \mid \forall \bar{x}.\, \varphi \in \mathsf{Q},\ \mathsf{dom}(\sigma) = \{\bar{x}\} \wedge \mathsf{ran}(\sigma) \subseteq \mathbf{T}(\mathsf{E}_i)\}$;
- $\mathsf{E}_0 = \mathsf{E}$, $\mathsf{E}_{i+1} \models \mathsf{E}_i \cup \mathsf{Q}_i$;

then $\mathsf{E} \cup \mathsf{Q}$ *is satisfiable in the empty theory with equality.*

Direct application at



- Quantifier-free solver enumerates assignments $\mathsf{E} \cup \mathsf{Q}$
- Instantiation module generates instances of Q

Enumerative instantiation

$\mathbf{u}(\mathsf{E}, \forall \bar{x}.\, \varphi)$:

1. Choose an ordering $\preceq$ on tuples of quantifier-free terms.
2. Return $\{\{\bar{x} \mapsto \bar{t}\}\}$ where $\bar{t}$ is a minimal tuple of terms w.r.t $\preceq$, such that $\bar{t} \in \mathbf{T}(\mathsf{E})$ and $\mathsf{E} \not\models \varphi\{\bar{x} \mapsto \bar{t}\}$, or $\emptyset$ if none exist.

- $\mathsf{E} = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $\mathsf{Q} = \{\forall x.\, P(x) \vee R(x)\}$
- $\mathbf{u}$ chooses an ordering on tuples of terms, say the lexicographic extension of $\preceq$ where $a \prec b \prec c$.
- Since E does not entail $P(a) \vee R(a)$, this strategy returns $\{\{x \mapsto a\}\}$.

u as an alternative for **m**

- Enumerative instantiation plays a similar role to MBQI
- It can also serve as a "completeness fallback" to **c** and **e**
- However, **u** has advantages over **m** for UNSAT problems
- Moreover it is significantly simpler to implement
  - No model building
  - No model checking

Example

$$\mathsf{E} = \{\neg P(a),\ R(b),\ S(c)\}$$
$$\mathsf{Q} = \{\forall x.\ R(x) \vee S(x),\ \forall x.\ \neg R(x) \vee P(x),\ \forall x.\ \neg S(x) \vee P(x)\}$$
$$\mathsf{M} = \left\{ \begin{array}{lcl} P^{\mathcal{M}} & = & \lambda x.\ \bot, \\ R^{\mathcal{M}} & = & \lambda x.\ \mathsf{ite}(x \simeq b,\ \top,\ \bot), \\ S^{\mathcal{M}} & = & \lambda x.\ \mathsf{ite}(x \simeq c,\ \top,\ \bot) \end{array} \right\}, \qquad a \prec b \prec c$$

φ	x s.t. $\mathcal{M} \not\models \varphi$	x s.t. $\mathsf{E} \not\models \varphi$	$\mathbf{m}(\mathsf{E}, \forall x.\ \varphi)$	$\mathbf{u}(\mathsf{E}, \forall x.\ \varphi)$
$R(x) \vee S(x)$	a	a	$\{\{x \mapsto a\}\}$	$\{\{x \mapsto a\}\}$
$\neg R(x) \vee P(x)$	b	a, b, c	$\{\{x \mapsto b\}\}$	$\{\{x \mapsto a\}\}$
$\neg S(x) \vee P(x)$	c	a, b, c	$\{\{x \mapsto c\}\}$	$\{\{x \mapsto a\}\}$

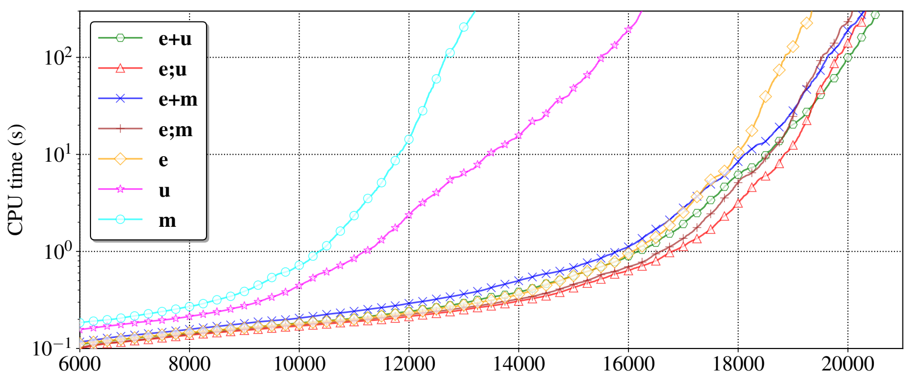
- **u** instantiates uniformly so that new terms are introduced less often
- **m** instantiates depending on how model was built
- Moreover, **u** leads to $\mathsf{E} \wedge \mathsf{Q}\{x \mapsto a\} \models \bot$
- **m** requires considering E' which satisfies E along the new instances

Implementation

Implementing enumerative instantiation efficiently depends on:

- Restricting enumeration space
- Avoiding entailed instantiations
- Term ordering to introduce new terms less often

CVC4 configurations on unsatisfiable benchmarks



- 42 065 benchmarks, being 14 731 from TPTP and 27 334 from SMT-LIB
- **e**+**u** stands for "interleave **e** and **u**", while **e**;**u** for "apply **e** first, then **u** if it fails"
- All CVC4 configurations have "**c**;" as prefix

Impact of u on satisfiable benchmarks

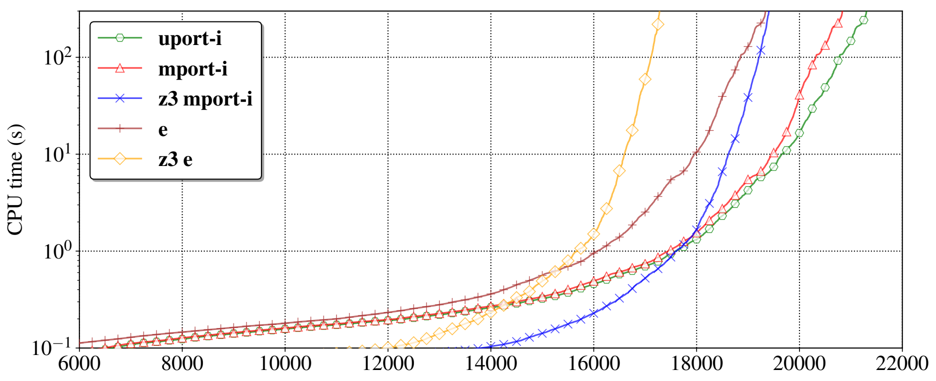
Library	#	**u**	**e;u**	**e+u**	**e**	**m**	**e;m**	**e+m**
TPTP	14731	471	492	464	17	930	808	829
UF	7293	39	42	42	0	70	69	65
Theories	20041	3	3	3	3	350	267	267
Total	42065	513	537	509	20	1350	1144	1161

- As expected, **m** greatly outperforms **u**
- Nevertheless **u** answers SAT *half as often* as **m** in empty theory
- Moreover, **u** solves 13 problems **m** does not

Impact of u on unsatisfiable benchmarks

- **u** solves $3\,043$ more benchmarks than **m**
- **u** solves $1\,737$ problems not solvable by **e**
- Combinations of **e** with **u** or **m** lead to significant gains
- **e+u** is best configuration, solving 253 more problems than **e+m** and $1\,295$ more than **e**
- Some benchmark families only solvable due to enumeration
- Overall the enumerative strategies lead to a virtual portfolio of CVC4 solving 712 more problems

Comparison against other instantiation-based SMT solvers



- Portfolios run without interleaving strategies (not supported by Z3)
- Z3 uses several optimizations for **e** not implemented in CVC4
- Z3 does not implement **c**

Restricting Enumeration Space

- Strengthened Herbrand Theorem allows restriction to $\mathbf{T}(\mathsf{E})$
- *Sort inference* [CS03] reduces instantiation space by computing more precise sort information
  - $\mathsf{E} \cup \mathsf{Q} = \{a \not\simeq b,\ f(a) \simeq c\} \cup \{P(f(x))\}$
    - $a, b, c, x : \tau$
    - $f : \tau \to \tau$ and $P : \tau \to$ Bool.
  - This is equivalent to $\mathsf{E}^s \cup \mathsf{Q}^s = \{a_1 \not\simeq b_1,\ f_{12}(a_1) \simeq c_2\} \cup \{P_2(f_{12}(x_1))\}$
    - $a_1, b_1, x_1 : \tau_1$
    - $c_2 : \tau_2$
    - $f_{12} : \tau_1 \to \tau_2$ and $P : \tau_2 \to$ Bool
  - **u** would derive e.g. $\{x \mapsto c\}$ for $\mathsf{E} \cup \mathsf{Q}$, while for $\mathsf{E}^s \cup \mathsf{Q}^s$ the instantiation $\{x_1 \mapsto c_2\}$ is not well-sorted.

Term Ordering

Instantiations are enumerated according to the order

$$(t_1, \ldots, t_n) \prec (s_1, \ldots, s_n) \quad \text{if} \quad \begin{cases} \max_{i=1}^n t_i \prec \max_{i=1}^n s_i, \text{ or} \\ \max_{i=1}^n t_i = \max_{i=1}^n s_i \text{ and} \\ \qquad (t_1, \ldots, t_n) \prec_{\text{lex}} (s_1, \ldots, s_n) \end{cases}$$

for a given order $\preceq$ on ground terms.

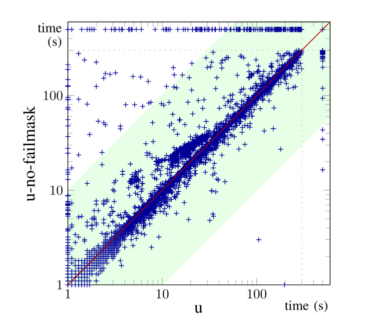
If $a \prec b \prec c$, then

$$(a, a) \prec (a, b) \prec (b, a) \prec (b, b) \prec (a, c) \prec (c, b) \prec (c, c)$$

- We consider instantiations with c only after considering all cases with a and b
- Goal is to introduce new terms less often
- Order on $\mathbf{T}(\mathrm{E})$ fixed for finite set of terms $t_1 \prec \ldots \prec t_n$
  - Instantiate in order with $t_1, \ldots, t_n$
  - Then choose new non-congruent term $t \in \mathbf{T}(\mathrm{E})$ and have $t_n \prec t$

Fair and Adventurous Enumeration of Quantifier Instantiations [JBFR21]

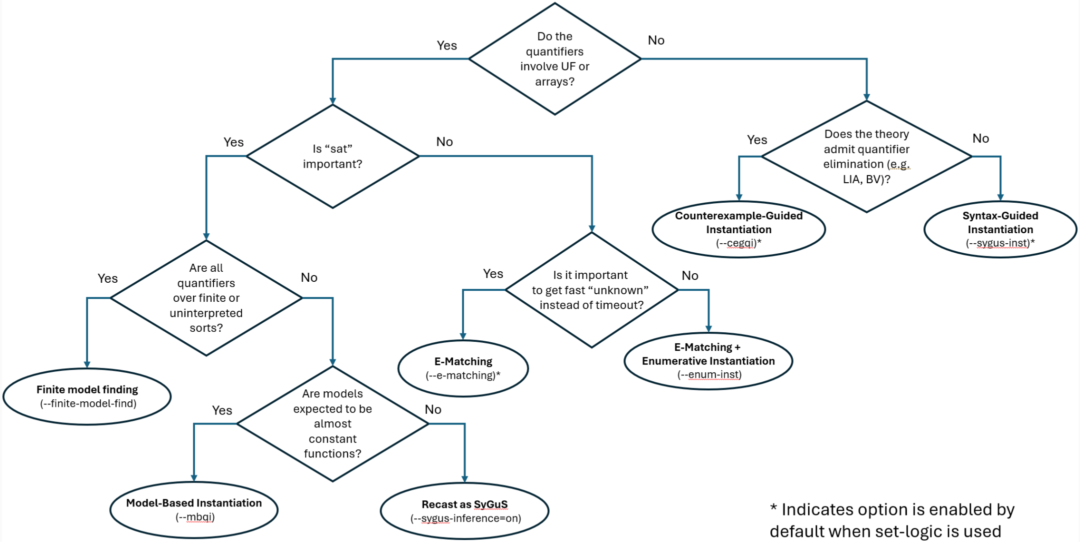
- Recently we tried a number of different orderings for tuples
  - Significant orthogonality, no clear winner
- In the process we improved the generation of instances in $\mathbf{u}$ to remove redundant tuples
  - A subset of the tuple is enough to lead to an entailed instance
  - A subset of the tuple is enough to lead an equivalent, modulo rewriting, instance



- UF, UFLIA, and UFNIA, totaling 31314 problems

So what quantifier handling techniques do I use if I have a quantified SMT problem?

A flowchat to pick quantifier handling strategies, at least in cvc5
(courtesy of Andy Reynolds)



* Indicates option is enabled by default when set-logic is used

Conclusion

- SMT solvers mainly use instantiation techniques to handle quantified formulas
- They are often incomplete, slow, or fragile
- But they also enable many successful use cases in e.g. software verification and automation for proof assistants
- There are many options, ask developers.

An introduction to SMT solving with quantifiers

Haniel Barbosa, Universidade Federal de Minas Gerais

SAT/SMT/AR Summer School

2024–06–27, LORIA–Inria, Université de Lorraine, Nancy, FR

CCFV calculus

$$\frac{E_\sigma \Vdash_E x \not\simeq y \wedge C}{E_\sigma \Vdash_E \bigvee_{[t],\,[t']\in E^{CC},\, E\models t\not\simeq t'} (x \simeq t \wedge y \simeq t' \wedge C)}\ \textsc{Dvar}$$

$$\frac{E_\sigma \Vdash_E x \not\simeq f(\bar{s}) \wedge C}{E_\sigma \quad \Vdash_E \quad \bigvee_{\substack{[t],\,[t']\in E^{CC},\\ E\models t\not\simeq t',\, f(\bar{t'})\in[t']}} (x \simeq t \wedge s_1 \simeq t'_1 \wedge \cdots \wedge s_n \simeq t'_n \wedge C)}\ \textsc{Dfapp}$$

$$\frac{E_\sigma \Vdash_E f(\bar{u}) \not\simeq g(\bar{s}_m) \wedge C}{E_\sigma \quad \Vdash_E \quad \bigvee_{\substack{[t],\,[t']\in E^{CC},\, E\models t\not\simeq t',\\ f(\bar{t})\in[t],\, g(\bar{t'}_m)\in[t']}} \begin{pmatrix} u_1 \simeq t_1 \wedge \cdots \wedge u_n \simeq t_n \ \wedge \\ s_1 \simeq t'_1 \wedge \cdots \wedge s_m \simeq t'_m \wedge C \end{pmatrix}}\ \textsc{Dgen}$$

$$\frac{E_\sigma \Vdash_E C_1 \vee C_2}{E_\sigma \Vdash_E C_1 \qquad E_\sigma \Vdash_E C_2}\ \textsc{Split} \qquad \frac{E_\sigma \Vdash_E \ell \wedge C}{E_\sigma \Vdash_E C}\ \textsc{Yield} \quad \text{if } E \models \ell$$

$$\frac{E_\sigma \Vdash_E \ell \wedge C}{E_\sigma \Vdash_E \bot}\ \textsc{Fail} \quad \text{if } \ell \text{ is ground and } E \not\models \ell$$

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